# OUTLINE OF GENERAL PHYSICS 

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(from an original by Graham Shaw)

## Introduction

An "outline" usually comprises a summary of important results, worked examples and some problems. Each section of this outline of general physics is devoted to a particular area of physics. Each section contains a few formulae which every physicist should know. These are intended as reminders only; the symbols are not defined but have their usual meanings. Then there is a set of questions chosen to cover the topics, taken from past third-year general papers. Where the answer can be given without revealing how to do the problem, then it is given.

There will be nine general-paper workshops in semester 6, from week 2 through week 10, each covering one section. You should try as many questions as possible from the relevant section before each workshop.

The questions marked with * are the minimum that you should expect to cover for the workshops, and these must be attempted in advance.

The 2008 third-year general paper is included, as section 9 . You should attempt this in full before the ninth and last workshop, in week ten. Also included are the Table of Physical Constants and Conversion Factors, which is issued with all physics examination papers, and a section on the "Mathematical Formulae You Should Know".

General physics exam papers for each of the last ten years are available on or through the teaching web, https://teachweb.ph.man.ac.uk/.

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## Newton's Laws

Difficulties in this area rarely arise from not knowing Newton's laws:

1. Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it;
2. The rate of change of momentum equals the total force on an object, $\boldsymbol{F}=\frac{\mathrm{d} p}{\mathrm{~d} t}$ ( $\boldsymbol{F}=m \boldsymbol{a}$ for constant mass);
3. For every action there is an equal and opposite reaction.

Many problems can be solved quickly and easily by applying conservation of mechanical energy (true when the forces are conservative) and in many cases conservation of momentum (true when there are no external forces).

## Newton's Laws for Rotation

We often use the equivalent of Newton's second law for rigid-body rotation. With

$$
L=\mathcal{I} \omega
$$

$$
\mathcal{I}=\sum_{i} m_{i} r_{i}^{2}
$$

$$
E_{L}=\frac{L^{2}}{2 \mathcal{I}}=\frac{1}{2} \mathcal{I} \omega^{2}
$$

we have Newton's second law for torque

$$
\mathcal{T}=\mathrm{d} \boldsymbol{L} / \mathrm{d} t=\mathcal{I} \mathrm{d} \omega / \mathrm{d} t
$$

Note that these equations are easy to remember. Apart from the one defining the moment of inertia, they are analogous to the ones for linear motion, with the substitutions:

$$
m \rightarrow \mathcal{I}, \quad v \rightarrow \boldsymbol{\omega}, \quad p \rightarrow \boldsymbol{L}, \quad \boldsymbol{F} \rightarrow \mathcal{T}
$$

Most of the time we shall only consider rotation around an axis of fixed direction, which can be treated as one-dimensional. In that case we drop the vector notation above.

## Damped Oscillators

The Equation of motion for a forced, damped simple harmonic oscillator is

$$
\ddot{x}+\gamma \dot{x}+\omega_{0}^{2} x=F / m
$$

For sinusoidal driving force $F(t)=F_{0} \cos \omega t$ the amplitude is:

$$
A(\omega)=\frac{F_{0} / m}{\left[\left(\omega_{0}^{2}-\omega^{2}\right)^{2}+(\gamma \omega)^{2}\right]^{1 / 2}}
$$

Critical damping occurs when $\gamma=2 \omega_{0}$. The full width of power curve at half maximum equals $\gamma$, and the $Q$-value is $Q=\omega_{0} / \gamma$.
The amplitude of the oscillations in the absence of forcing is

$$
A(t)=A_{0} \exp (-\gamma t / 2)
$$

### 1.1 Selected problems

1. An escalator inclined at $45^{\circ}$ is moving up at a speed of $1 \mathrm{~m} \mathrm{~s}^{-1}$. A man who weighs 70 kg walks up it at a speed relative to the escalator of $0.5 \mathrm{~m} \mathrm{~s}^{-1}$. At what rate is he doing work? Answer: 243 W
2.     * A bullet travelling horizontally strikes a large block of wood, suspended by thin wires, and remains stuck in it. The impact causes the block to swing upward to a height of 20 cm . If the bullet has a mass of 10 g and the block's mass is 4 kg , what was the speed of the bullet immediately before impact?
Answer: $793 \mathrm{~m} \mathrm{~s}^{-1}$
3. A projectile explodes into two pieces at the top of its trajectory, a distance $L$ measured horizontally from its launch point. The two resulting fragments have masses $\frac{1}{4}$ and $\frac{3}{4}$ of the original mass and emerge horizontally from the explosion with the small fragment landing back at the original launch point. How far from the original launch point does the larger fragment land?
Answer: 8L/3
4.     * A piece of space debris, initially at rest, falls radially towards the Moon. It is initially at twice the Moon's radius from the centre of the Moon. Calculate the velocity of the debris when it hits the Moon's surface.
The mass and radius of the Moon are $7.35 \times 10^{22} \mathrm{~kg}$ and $1.74 \times 10^{6} \mathrm{~m}$ respectively.
Answer: $1.68 \times 10^{3} \mathrm{~m} \mathrm{~s}^{-1}$
5. Gravity Probe B has just been launched into a circular orbit round the Earth at a height of 650 km . Calculate the time it takes to complete one orbit. Answer: 5900 s
6. An intrepid Australian digs a hole through the centre of the Earth from Queensland to Spain and then drops a small object of mass $m$ down the hole. Assuming that the Earth is of uniform density, calculate the gravitational force acting on the mass at a distance $R$ from the centre of the Earth. Describe the motion of the mass.
Answer: $G m R M_{\mathrm{E}} / R_{\mathrm{E}}^{3}$, where $M_{\mathrm{E}}$ and $R_{\mathrm{E}}$ are mass and radius of the Earth.
7. A neutron star has a moment of inertia of $10^{38} \mathrm{~kg} \mathrm{~m}^{2}$ and has a rotation period of 2 milliseconds. The period is increasing by $10^{-13} \mathrm{sec} / \mathrm{sec}$. What is the rate of loss of kinetic energy? Answer: $\left(\pi^{2} / 2\right) 10^{34} \mathrm{~J} \mathrm{~s}^{-1}$
8.     * A massless rope is wrapped several times around a solid cylinder of radius $\mathrm{R}=20 \mathrm{~cm}$ and mass $\mathrm{M}=20 \mathrm{~kg}$, which is at rest on a horizontal surface, as shown below.


Someone pulls 1 m of the rope with a constant force of 100 N , setting the cylinder in motion. Assuming that the rope neither stretches nor slips and that the cylinder rolls without slipping, what is the final angular velocity of the cylinder and the speed at its surface?
The moment of inertia of a cylinder of mass $M$ and radius $R$ is $M R^{2} / 2$.
Answer: $\omega=18.4 \mathrm{rad} / \mathrm{s}, v=3.65 \mathrm{~m} \mathrm{~s}^{-1}$
9. A bowling ball is rolling without slipping up an inclined plane. As it passes a point $O$ it has a speed of $2.00 \mathrm{~ms}^{-1}$ up the plane. It reaches a vertical height $h$ above $O$ before momentarily stopping and rolling back down. Determine the value of $h$.
If a hollow ball, composed only of a thin shell, underwent similar motion also moving at a speed of $2.00 \mathrm{~ms}^{-1}$ at $O$, would the height it reached be less than, equal to, or greater than that of the solid bowling ball? Give reasons for your answer.
The moment of inertia of a solid sphere of mass $M$ and radius $R$ is $\mathcal{I}=2 M R^{2} / 5$. Answer: $h=0.29 \mathrm{~m}$
10. * It is proposed to fill the tender of a moving steam engine with coal dropping vertically from a hopper. What is the extra tractive force which must be applied to the tender to maintain constant velocity if it is to be loaded with 10 tonnes of coal in 2 s and proceeds uniformly for 10 m during this time? Neglect any frictional effects.
Answer: $25 \times 10^{3} \mathrm{~N}$
11. * A ball of mass $m_{1}$ strikes a stationary ball of mass $m_{2}$ with velocity $v_{1}$. What is the maximum velocity which the second ball could acquire? Answer: $2 m_{1} v_{1} /\left(m_{1}+m_{2}\right)$
12. * A vertical U-tube of uniform cross section open at both ends contains a liquid which fills in total a length $l$ of the tube. Calculate the period of oscillations of the liquid, if the liquid in one of the arms of the U-tube is pushed down a small distance and then released. Answer: $2 \pi \sqrt{l / 2 g}$
13. A sieve is moving up and down with an amplitude of $x_{0}=50 \mathrm{~mm}$. If sand grains lying on the sieve are supposed to become detached from the sieve during the harmonic oscillation, what is the maximum vibration period? Answer: 0.45 s
14. * The figure below shows the response of a forced oscillator as a function of driving frequency. Estimate the decay time for the amplitude of free oscillations Answer: 1.3 s

15. A piece of thin copper rod, mass $m$, is formed into a square of side length $l$, and suspended vertically on a wire of torsional constant $\kappa$. The moment of inertia of the square for rotations about the vertical axis is $m l^{2} / 6$. Find the period of rotational oscillations.
Answer: $T=2 \pi \sqrt{\frac{m l^{2}}{6 \kappa}}$.
16. The side of a thin disk of mass $m$ and radius $r$, pivoted at its centre, is attached to the end of an unstretched spring of force constant $k$, as shown in the diagram below. The disk is rotated slightly and released. Calculate the period of small oscillations.
Answer: $\omega=\sqrt{2 k / M}$


## 2 DIMENSIONS, ASTRONOMY and DATA ANALYSIS

## The method of dimensions

All equations in physics must be dimensionally consistent (i.e. both sides must have the same units). It is sometimes possible to deduce an approximate formula, usually up to a multiplicative constant, by combining the relevant quantities in such a way as to get the right units. This can be used in practice by assuming that a physical quantity $Z$ that depends on a number of other quantities $A, B, C, \ldots$ must obey a power law to make the dimensions on the left and right-hand sides be equal. We thus set $Z=c A^{\alpha} B^{\beta} C^{\gamma} \ldots$, with $c$ a dimensionless constant. We solve by equating the powers on the basic units on left- and right-hand sides (usually length, time and mass are sufficient; occasionally we need to consider temperature and current; the candela is not usually required). If there is one unique set of powers $\alpha, \beta, \gamma$ etc. that satisfy the equality of dimensions, then we have determined a suitable functional form of $Z$. The method fails to provide a unique solution if there is a dimensionless quantity of the form $A^{a^{\prime}} B^{\beta^{\prime}} C^{\gamma^{\prime}} \ldots$.
Note that temperature usually enters into physical quantities as $k T$ or equivalently $R T$, which both have the dimensions of energy. The SI units of viscosity are $\mathrm{kg} \mathrm{m}^{-1} \mathrm{~s}^{-1}$. Surface tension may be considered as force per unit length or energy per unit area, and has units of $\mathrm{kg} \mathrm{s}^{-2}$.

## Introductory Astronomy

Kepler's third law: $T^{2}=k R^{3}$ ( $k=1$ if $T$ is in years and $R$ is in astronomical units).
Blackbody radiation: $P=\sigma T^{4}$ is power emitted per unit area. (Stefan's law)
Difference in magnitude: $\Delta m=2.5 \log _{10} R$, where $R$ is the brightness ratio.
Non-relativistic Doppler shift: $\Delta \lambda / \lambda=v / c$.
Reduced mass for a 2-body system: $\mu=m_{1} m_{2} /\left(m_{1}+m_{2}\right)$.
Orbit around common centre of mass: $m_{1} R_{1}=m_{2} R_{2}$.

## Data Analysis for Laboratory

If $f=f(x, y, z, \ldots)$ and $x, y, z, \ldots$ represent uncorrelated measurements then

$$
\sigma_{f}^{2}=\left(\frac{\partial f}{\partial x}\right)^{2} \sigma_{x}^{2}+\left(\frac{\partial f}{\partial y}\right)^{2} \sigma_{y}^{2}+\left(\frac{\partial f}{\partial z}\right)^{2} \sigma_{z}^{2}+\ldots
$$

Special cases:
If $f=x \pm y$ then $\sigma_{f}^{2}=\sigma_{x}^{2}+\sigma_{y}^{2}$;
If $f=x^{m} y^{-n}$ then $\left(\frac{\sigma_{f}}{f}\right)^{2}=m^{2}\left(\frac{\sigma_{x}}{x}\right)^{2}+n^{2}\left(\frac{\sigma_{y}}{y}\right)^{2}$.
For a set of measurements $x_{i}$ each with the same error $\sigma$, the best estimate of the true value is the mean

$$
\bar{x}=\frac{1}{n} \sum x_{i}
$$

The error on the mean is

$$
\sigma_{\bar{x}}=\frac{\sigma}{\sqrt{n}}
$$

The distribution of the measured values $x_{i}$ has variance:

$$
s^{2}=\frac{1}{n} \sum\left(x_{i}-\bar{x}\right)^{2}
$$

and $\sqrt{\frac{n}{n-1}} \times s$ is the best estimate of $\sigma$, the error on each measurement.
The Binomial distribution gives the probability of $n$ "successes" in $N$ trials:

$$
\mathrm{P}(n, N)=p^{n}(1-p)^{N-n} \frac{N!}{n!(N-n)!},
$$

where $p$ is the probability of success in one trial.
The Poisson distribution is the limit of the binomial distribution for $p \rightarrow 0$ and $N \rightarrow \infty$. If $\mu$ is the expected number of "successes", then

$$
\mathrm{P}(n)=\frac{\mu^{n} \mathrm{e}^{-\mu}}{n!}
$$

The standard deviation of the Poisson distribution is $\sqrt{\mu}$. If $\mu$ is unknown, the best estimate of the standard deviation is $\sigma_{n}=\sqrt{n}$.

### 2.1 Selected problems

1.     * By dimensional arguments show that the speed of waves on a deep body of liquid is independent of the liquid density if the waves are long enough to be controlled by gravity, but not if they are so short as to be controlled by surface tension.
2. After being deformed, a spherical drop of an incompressible liquid will execute periodic vibrations. Use the method of dimensions to obtain an expression for the frequency of these vibrations in terms of the physical properties of the drop.
3.     * The upthrust force $F$ on an aeroplane wing of fixed cross-sectional shape is proportional to the length of the wing. Use a simple dimensional argument to show how it depends on the width of the wing $d$, the density of air $\rho$, and the velocity of the aeroplane $v$.
4. Estimate the pressure at which a gas of argon atoms, at a temperature of 300 K , will begin to show deviations from the ideal gas behaviour due to the finite size of the atoms. [Use the method of dimensions to get an order of magnitude answer.] Answer: Of order $10^{9} \mathrm{~Pa}$
5.     * The filament of an incandescent light bulb emits power per unit area at a rate proportional to $\sigma T^{4}$, where $\sigma$ is the Stefan-Boltzmann constant and $T$ is the absolute temperature. If the current in the bulb changes so that its temperature falls from 3000 K to 2900 K , estimate by what fraction the total power output is reduced? Assume that the area remains the same.
6. Not suitable for some joint honours programmes Draw a Hertzsprung-Russell diagram, clearly labelling the axes. Show the main sequence and the regions where red giants, blue super-giants and white dwarfs are located.
7.     * Not suitable for some joint honours programmes.

A type-II Cepheid variable star is observed in a distant galaxy. It has a period of 100 days and a peak brightness 10 magnitudes less than the brightness of a 100-day period, type-II Cepheid variable observed in the Large Magellanic Cloud (LMC). Given that the LMC lies at a distance of 150,000 light years from our Sun, calculate the distance to the distant galaxy. Answer: $1.5 \times 10^{7}$ light years
8. Venus orbits the Sun every 224.7 days. Assuming that the orbits of the Earth and Venus are both circular, calculate the round-trip travel time of a radar pulse reflected from Venus at the time of the Venus transit of the Sun in June 2004.
Answer: 275 s
9. The black hole remnant of a star has a mass of 12 Solar Masses. Calculate its Schwarzschild radius.
Answer: 36 km
10. * The radial velocity of a 1 solar mass star is seen to vary sinusoidally by $\pm 45 \mathrm{~m} \mathrm{~s}^{-1}$ with a period of 1095 days. Assuming that the Earth lies in the plane of the orbit of the object causing this stellar motion, calculate the object's mass in solar masses and its distance from the star in astronomical units (AU).
Answer: $2 \times 10^{-3}$ solar masses and 2.1 AU
11. * A surveyor measures the angle of elevation of a mountain top from the horizontal to be $22.2 \pm 0.1^{\circ}$ and the direct distance to the top of the mountain to be $1536 \pm 3 \mathrm{~m}$. Calculate the vertical height of the mountain above the surveyor, and the error in that height. Answer: $580.4 \pm 2.7 \mathrm{~m}$
12. * Two experimental methods, each free from systematic errors, were used to measure the height, $h$, of a building. The mean values obtained were $h_{A}=(71 \pm 4) \mathrm{m}$ from 64 measurements using method $A$ and $h_{B}=(68 \pm 6) \mathrm{m}$ from 16 measurements using method $B$. Show which method has the higher precision. How many extra readings must be taken using method B to ensure that the final results for $h_{A}$ and $h_{B}$ have the same precision? Answer: 20
13. A sample contains the measurements $3.4,3.6,3.7,3.8$. Estimate the standard deviation of the parent population. Answer: 0.17
14. I suspect that a coin is biased, turning up heads on 45 percent of tosses. How many trials would I need to confirm with reasonable certainty that it is biased? Answer: Of order 1000.
15. Very low intensity pulses of light have a Poisson distribution in the number of photons with a mean number of photons/pulse of 10. They are detected with photomultipliers with a photoelectric efficiency of 25 per cent. What is the probability of detecting one pulse of light if the photomultiplier is sensitive to single photo-electrons? The Poisson formula for the probability of $m$ photons, when the mean is $\bar{n}$, is: $p_{m}=\bar{n}^{m} \mathrm{e}^{-n} / m!$.
Answer: 0.92

## 3 RELATIVITY AND PARTICLE PHYSICS

The trickiest bit is remembering the factors of $c$, so always check them by dimensions, remembering that $E, p c$, and $m c^{2}$ all have dimensions of energy.

Lorentz Transformation

$$
\begin{array}{ll}
x^{\prime}=\gamma(x-v t), & \\
y^{\prime}=y, & \text { where } \gamma \equiv \frac{1}{\left(1-v^{2} / c^{2}\right)^{1 / 2}} \\
z^{\prime}=z, \\
t^{\prime}=\gamma\left(t-v x / c^{2}\right) . &
\end{array}
$$

$$
t=\gamma t_{0}, \quad L=L_{0} / \gamma
$$

Velocity addition

$$
\begin{aligned}
u_{x}^{\prime} & =\frac{u_{x}-v}{1-u_{x} v / c^{2}}, \\
u_{y}^{\prime} & =\frac{\gamma u_{y}}{1-u_{x} v / c^{2}}, \quad \text { and similarly for } u_{z} .
\end{aligned}
$$

Energy and momentum
implying

$$
E^{2}=(p c)^{2}+\left(m c^{2}\right)^{2} \quad \text { and } \quad \gamma=E /\left(m c^{2}\right)
$$

By definition the kinetic energy is $E-m c^{2}$.
Invariant mass: a Lorentz invariant, which is conserved in collisions and decays

$$
W^{2}=\left(\sum_{i} E_{i}\right)^{2}-\left(\sum_{i} p_{i} c\right)^{2}
$$

Exponential decay law

$$
N(t)=N(0) \mathrm{e}^{-t / \tau}=N(0)(1 / 2)^{t / \tau_{1 / 2}},
$$

where $\tau$ is the "lifetime" (i.e. mean lifetime) and $\tau_{1 / 2}=\tau \ln 2$ is the half-life.
Photons and other massless particles

$$
E=p c=h c / \lambda
$$

Compton Scattering:

$$
\lambda^{\prime}=\lambda+\frac{h c}{m c^{2}}(1-\cos \theta)
$$

Particles You should be familiar with the quantum numbers, quark content (where appropriate) and reactions (strong, EM, weak) of the following particles: $\mathrm{p}, \mathrm{n}, \Lambda, \Sigma, \Omega$, pions, kaons, e, $\mu$, $\tau$ and neutrinos.

### 3.1 Selected problems

1. A rocket is launched at a speed of $0.5 c$ from the earth. After one year measured by a timing mechanism in the rocket a signal is sent back to earth. How long after the launch is the signal received?
Answer: $\sqrt{3}$ years
2.     * A beam of kaons is aimed at a target 10 m away. If the kaons have a momentum of 5 $\mathrm{GeV} / c$, calculate the fraction of them that reach the target without decaying. (Kaon mean lifetime $=1.2 \times 10^{-8} \mathrm{~s}$ : kaon rest mass $=500 \mathrm{MeV} / \mathrm{c}^{2}$ ) Answer: 0.76
3. A radioactive source of half-life 1 year is found to emit $10^{6}$ particles per second. It is then sent by rocket at a speed of $0.6 c$ on a round trip to a manned space station; it arrives back exactly one year later. What is the disintegration rate when it arrives back? Answer: $5.7 \times 10^{5} \mathrm{~s}^{-1}$
4.     * Two protons, each with kinetic energy of 1 GeV , are made to collide head-on. Calculate their relative velocity before collision. Answer: 0.991c
5.     * A photon and a particle of mass $10 \mathrm{eV} / \mathrm{c}^{2}$ are produced simultaneously in a galaxy 150,000 light years away. Both particles have energy 10 MeV . Estimate the difference in arrival time between the two particles.
Answer: 2.4 s
6. A 1.022 MeV photon is Compton scattered through $90^{\circ}$ by an electron and emerges with an energy of 0.341 MeV . What is the speed of the recoiling electron? Answer: 0.903c
7.     * A neutral $\pi^{0}$ meson (rest mass $135 \mathrm{MeV} / c^{2}$ ) is observed to decay in flight into two photons each with an energy of 80 MeV . Calculate the angle between the emitted photons. Answer: $115^{\circ}$
8. A stationary $\Sigma^{+}$particle decays into a proton and a $\gamma$-ray of energy 224 MeV . What is the recoil kinetic energy of the proton and the rest mass of the $\Sigma^{+}$particle? Answer: 26 MeV ; $1188 \mathrm{MeV} / \mathrm{c}^{2}$
9. At HERA, a 920 GeV proton beam collided with a 27.5 GeV electron beam. What is the heaviest particle which could, theoretically, have been created there? Answer: 318 GeV
10.     * A beam of protons is incident on a hydrogen target. Calculate the minimum energy of the incident proton if a $\mathrm{K}^{-}$particle is to be observed among the reaction products. (The proton mass is $938 \mathrm{MeV} / c^{2}$; the charged kaon mass is $494 \mathrm{MeV} / c^{2}$.) Answer: 3.43 GeV
11.     * Classify the following reactions into strong, EM, weak or forbidden, explaining your reasons:

$$
\begin{aligned}
& \mathrm{p}+\overline{\mathrm{p}} \rightarrow \mathrm{~K}^{+}+\pi^{-}+\Lambda \\
& \Lambda \rightarrow \mathrm{K}^{-}+\mathrm{p} \\
& \mathrm{~K}^{-}+\mathrm{p} \rightarrow \Lambda+\gamma \\
& v_{\mu}+\mathrm{p} \rightarrow \mu^{-}+\pi^{+}+\pi^{0}+\mathrm{p} \\
& \mathrm{~K}^{+} \rightarrow \pi^{0}+\pi^{+}+\gamma
\end{aligned}
$$

(Masses: $m_{\mathrm{p}}=938 \mathrm{MeV} / c^{2} ; m_{\Lambda}=1116 \mathrm{MeV} / c^{2} ; m_{\pi^{ \pm}}=140 \mathrm{MeV} / c^{2} ; m_{\mathrm{K}^{ \pm}}=494 \mathrm{MeV} / c^{2}$; $m_{\mu}=106 \mathrm{MeV} / c^{2}$ )
Partial answer: One EM, two weak and two forbidden.

## 4 QUANTUM MECHANICS, ATOMS AND NUCLEI

Photoelectric effect : Maximum energy of emitted electrons is $E_{\max }=h c / \lambda-\phi$ where $\lambda$ is the
wavelength of incident photons and $\phi$ is the work function of the surface.
De Broglie relations: $p=h / \lambda=\hbar k$ and $E=h v=\hbar \omega$.
Momentum operator, eigenstates (one dimension):

$$
\hat{p}=-i \hbar \frac{d}{d x}, \quad \psi_{p}=\mathrm{e}^{i k x}=\mathrm{e}^{i p x / \hbar}
$$

Energy levels for simple harmonic oscillator:

$$
\epsilon_{n}=\left(n+\frac{1}{2}\right) \hbar \omega,
$$

where $\omega=(k / m)^{1 / 2}$ and $k$ is a constant analogous to the classical spring constant.
Energy levels for H -like ions:

$$
\epsilon_{n}=-Z^{2} R_{\infty} / n^{2}
$$

The formula and value for the Rydberg constant are given in the Table of Constants
The single electron energy levels in atoms, in order of increasing energy, are
$1 \mathrm{~s}, 2 \mathrm{~s}, 2 \mathrm{p}, 3 \mathrm{~s}, 3 \mathrm{p}, 4 \mathrm{~s}, 3 \mathrm{~d}, 4 \mathrm{p}, \ldots$.
Expectation values (for normalised wavefunction $\psi$ ):

$$
\bar{O}=\int \psi(x)^{*} \hat{O} \psi(x) d x
$$

First order Perturbation theory (for normalised wavefunction $\psi_{0}$ ):

$$
\Delta E=\int \psi_{0}(x)^{*} \Delta V(x) \psi_{0}(x) d x
$$

Natural line width:

$$
\Gamma=\Delta E=\hbar / \tau .
$$

Heisenberg Uncertainty Principle:

$$
\Delta x \Delta p \geq \hbar / 2
$$

where

$$
(\Delta p)^{2}=\overline{p^{2}}-\bar{p}^{2}, \quad \text { etc. }
$$

For bound states $\bar{p}=0$ so that $(\Delta p)^{2}=\overline{p^{2}}$. For simple attractive potentials, the uncertainties are near the minimum allowed by the Uncertainty Principle, so:

$$
\overline{p^{2}}=(\Delta p)^{2} \approx\left(\frac{\hbar}{\Delta x}\right)^{2}
$$

Angular momentum The same rules apply to all angular momentum vector operators $\boldsymbol{X}=\boldsymbol{J}, \mathbf{L}$,
$S, j, l, s$. They are:
(a) Angular momentum eigenstates with quantum numbers $X, m_{X}$ have

$$
|\boldsymbol{X}|=\hbar \sqrt{X(X+1)}, \quad X_{z}=m_{X} \hbar
$$

where the allowed values of $m_{X}=X, X-1, \ldots \ldots,-X$.
(b) Addition of angular momentum. If $\boldsymbol{X}=\boldsymbol{X}_{1}+\boldsymbol{X}_{2}$ the allowed values for the quantum number $X$ are

$$
X=X_{1}+X_{2}, X_{1}+X_{2}-1, \ldots,\left|X_{1}-X_{2}\right|
$$

Spectroscopy Students should be familiar with the spectroscopic notation ${ }^{2 S+1} L_{J}$ for atomic states.

### 4.1 Selected problems

1. Photoelectrons with a maximum energy of 0.4 eV are observed when light of wavelength 450 nm is incident on a surface. Calculate the maximum wavelength of light which will produce photoelectrons from this surface.
Answer: 530 nm
2.     * An atomic state has an emission line of wavelength $6 \times 10^{-7} \mathrm{~m}$ and natural width $10^{-13} \mathrm{~m}$. Estimate its lifetime. Answer: $2 \times 10^{-9} \mathrm{~s}$
3.     * Estimate, using the Uncertainty Principle, the kinetic energy of an electron if it were bound in the nucleus.
Answer: $\sim 200 \mathrm{MeV}$ for $R \sim 1 \mathrm{fm}$
4.     * A muon is a particle very similar to an electron but with mass $105.6 \mathrm{MeV} / c^{2}$, and a muonic atom is the bound state of a muon and a proton. Calculate the binding energy of the ground state of a muonic atom
Answer: 2.53 keV
5. Explain quantitatively how the wavelength for the $n=2$ to $n=1$ transition for deuterium differs from that for hydrogen, ignoring hyperfine structure. The masses of $\mathrm{H}^{1}, \mathrm{H}^{2}$ and e are $1.00728,2.01355$ and $5.58 \times 10^{-4}$ amu respectively.
6. A lead target (atomic number $Z=82$ ) is bombarded by an electron beam to produce $X$ rays. Estimate the minimum electron kinetic energy required to eject an electron from the innermost shell
Answer: 90 keV
7. A particle is confined by the one-dimensional square-well potential

$$
V(x)= \begin{cases}-V_{0} & -a<x<a \\ 0 & \text { otherwise }\end{cases}
$$

Sketch the wave functions both inside and outside the well for the three lowest energy levels, assuming that they are bound states.
8. The normalised wavefunction for the 1 s electron in the hydrogen atom is

$$
\psi=\frac{1}{\sqrt{\pi} a_{0}^{3 / 2}} \mathrm{e}^{-r / a_{0}}
$$

What is the mean value of $\left(1 / r^{2}\right)$ in terms of the Bohr radius $a_{0}$ ?
Answer: $2 / a_{0}^{2}$
9. * The normalised energy eigenfunctions and corresponding eigenvalues for a particle in a potential well are $\phi_{n}(x)$ and $E_{n}$, where $n$ is the quantum number. At time $t=0$ the state of a particle is

$$
\Phi(x, 0)=\frac{3}{5} \phi_{1}(x)+\frac{4}{5} \phi_{2}(x)
$$

Write down the expression for $\Phi(x, t)$ for the time-dependent wavefunction. What results could you expect to get for a measurement of the energy, and what are their probabilities?
10. An electron is described by the wavefunction

$$
\psi=A \sin (2 \pi x / L)
$$

where $L=3 \times 10^{-8} \mathrm{~m}$. What are the possible outcomes of a measurement of the electron's x-component of momentum?
Answer: $\pm 41.3 \mathrm{eV} / \mathrm{c}$
11. A particle is in the ground state in a one-dimensional box given by the potential

$$
V(x)= \begin{cases}0 & 0<x<a \\ \infty & \text { otherwise }\end{cases}
$$

A small perturbation $V=V_{0} x / a$ is now introduced. Show, correct to first order in perturbation theory, that the energy change in the ground state is $V_{0} / 2$.
The normalised wave functions of $V(x)$ are $\Psi_{n}=\sqrt{\frac{2}{a}} \sin \left(\frac{n \pi x}{a}\right)$
12. Give a brief explanation of the origin of the splitting of the sodium $D$ lines.
13. ${ }^{*}$ A $p$ state in a hydrogen-like atom is found to be split into two states by the spin-orbit coupling $V_{l s}=\lambda l$.s. Calculate the total angular momentum and energy shift of each of the two levels.
Answer: $\lambda \hbar^{2} / 2(j=3 / 2) ;-\lambda \hbar^{2}(j=1 / 2)$
14. Draw a clearly labelled graph indicating the main features of the dependence of the average binding energy per nucleon of the stable nuclei as a function of their mass number.
15. * The magnitude of the total angular momentum of a particle is given by $|\boldsymbol{L}|=\sqrt{6} \hbar$. Determine the possible values that could be obtained in a measurement of the $z$-component of term.
$L$
16. * A spin $\frac{1}{2}$ particle is observed (for example, in a Stern-Gerlach apparatus) to be in the state $s_{z}=\frac{\hbar}{2}$. The particle then passes through an apparatus that measures $s_{y}$ and is found to be in the state with $s_{y}=-\frac{\hbar}{2}$. What was the probability of obtaining that result?
The particle then passes through a further apparatus that measures $s_{z}$. What are the possible results of this measurement, and with what probability will they be obtained?

## 5 GASES, LIQUIDS and SOLIDS

## Boltzmann distribution

$$
p_{r} \propto \mathrm{e}^{-\epsilon_{r} / k T}
$$

Maxwell-Boltzmann speed distribution

$$
f(v) \propto v^{2} \exp \left(-\frac{m v^{2}}{2 k T}\right)
$$

Equipartition of energy

Bragg's law

$$
E=\frac{1}{2} k T \text { per degree of freedom. }
$$

$n \lambda=2 d \sin \theta$.
Mean free path The average fraction of particles travelling a distance $x$ without undergoing collisions is

$$
\frac{N(x)}{N(0)}=\mathrm{e}^{-x / \lambda}
$$

where the mean free path, for number density $n$, is:

$$
\lambda=\frac{1}{\sqrt{2} n \sigma}
$$

Ideal gases

$$
P V=n R T=N k T \quad \text { and } \quad E=E(T)
$$

where $n$ is number of moles and $N$ is number of particles. The energy is a function of temperature only, independent of pressure. In addition, the difference of specific heats is

$$
C_{P}-C_{V}=n R
$$

and

$$
P V^{\gamma}=\text { constant }
$$

for an adiabatic expansions where $\gamma \equiv C_{P} / C_{V}$.
STP: Standard temperature and pressure are defined as $T=0^{\circ} \mathrm{C}$ and $p=1 \mathrm{~atm}$.
Note: The symbol $n$ has been used with three different meanings in the equations above!

### 5.1 Selected problems

1.     * Estimate the mean free path and typical speed of atoms in a helium atmosphere at standard temperature and pressure. Answer: $\lambda \sim 8 \times 10^{-7} \mathrm{~m}(d=1 \AA)$ and speed $\sim 1.3 \times 10^{3} \mathrm{~m} \mathrm{~s}^{-1}$
2. Estimate the mean free path of argon atoms when the gas is at 300 K and 2 atmospheres pressure. (The diameter of an argon atom is $4 \times 10^{-10} \mathrm{~m}$ ). Answer: $3 \times 10^{-8} \mathrm{~m}$
3.     * Derive an expression for the most probable speed of a molecule in a sample of gas at temperature $T$.
Answer: $\sqrt{2 k T / m}$
4. In a cloud chamber for photographing the tracks of $\alpha$-particles the temperature of the air is $10^{\circ} \mathrm{C}$. If its volume is increased in the ratio 1.375 to 1 by a rapid expansion, calculate the final temperature of the air (the ratio of specific heats of air is 1.41 ).
Answer: 248 K
5. A skyscraper is 400 m high. Estimate the atmospheric pressure at the top of the skyscraper assuming that the building has a uniform temperature of $20^{\circ} \mathrm{C}$.
Answer: $0.98 \times 10^{5} \mathrm{~Pa}=0.96$ atmosphere
6.     * In a large underground cave, the concentration of helium in the air at the base is measured to be $1.00 \times 10^{-4}$ (this is the ratio of the number of He atoms to the total number of molecules). If the air in the cave is at a constant temperature of $10^{\circ} \mathrm{C}$, what is the concentration of helium at the top of the cave, 300 m higher?
Answer: $1.03 \times 10^{-4}$
7.     * Explain what is meant by the principle of equipartition of energy. Use it to determine the heat capacity $C_{V}$ of a diatomic gas of $N$ molecules.
The molar specific heat $c_{V}$ of hydrogen at a temperature of 55 K is measured to be approximately $12.5 \mathrm{~J} \mathrm{~mol}^{-1} \mathrm{~K}^{-1}$, but at a temperature of 320 K is found to equal approximately $20.8 \mathrm{~J} \mathrm{~mol}^{-1} \mathrm{~K}^{-1}$. Explain these results in the light of those predicted by equipartition.
8.     * The vibrational levels of carbon monoxide are spaced by $6.42 \times 10^{13} \mathrm{~Hz}$. Would you expect vibrational excitations to make an important contribution to the specific heat of CO at room temperature?
Answer: $T_{v} \sim 3000 \mathrm{~K}$ i.e. negligible at room temperature
9. What features of the interatomic potential energy curve are responsible in non-ionic solids for (i) the latent heat of sublimation (ii) thermal expansion?
10. The latent heat of vaporisation of carbon tetrachloride is $3.210^{4} \mathrm{~J} \mathrm{mo}^{-1}$, Estimate the binding energy between pairs of molecules in the liquid state. Answer: Approximately 0.07 eV
11. Estimate the molar heat capacity of argon gas at constant pressure. Comment on any difference between your estimate and the measured value of $21.13 \mathrm{~J} \mathrm{~K}^{-1}$ at room temperature and pressure.
12.     * X-rays of energy 25 keV show a first order Bragg reflection from a crystal at an incident angle of $10^{\circ}$ to the reflecting plane. Neutrons of energy 0.025 eV are Bragg reflected from the same crystal planes. For what angle does this reflection take place? Answer: $39.3^{\circ}$.
13. X-rays with a wavelength of 0.16 nm are reflected off a crystal of silicon. The first interference peak is observed at an angle of $36^{\circ}$ to the normal to the crystal plane. Find the spacing of the atoms in silicon.
Answer: $1.0 \times 10^{-10} \mathrm{~m}$

6 THERMAL AND STATISTICAL PHYSICS

Some relevant formulae, e.g. the ideal gas laws, have been given under "Gases, Liquids and Solids".

First law of thermodynamics

$$
\Delta E=Q+W
$$

Fundamental thermodynamic identity (hydrostatic systems)]

$$
\mathrm{d} E=T \mathrm{~d} S-P \mathrm{~d} V
$$

Reversible processes

$$
\mathrm{d} Q=T \mathrm{~d} S, \quad \mathrm{~d} W=-P \mathrm{~d} V
$$

Ideal heat engines

Isolated systems (fixed $E, V$ ).

$$
\frac{Q_{1}}{T_{1}}=\frac{Q_{2}}{T_{2}}
$$

$$
S=k \ln \Omega \quad, \quad \Delta S \geq 0
$$

Canonical systems (fixed $T, V$ )]
Partition function: $Z=\sum \mathrm{e}^{-\epsilon_{i} / k T}$
Boltzmann distribution: $p_{r}=\frac{\mathrm{e}^{-\beta \epsilon_{r}}}{\mathrm{Z}}, \quad \beta \equiv \frac{1}{k T}$.
Mean energy: $E=-\left(\frac{\partial \ln Z}{\partial \beta}\right)_{V}^{Z}$
Free energy (Helmholtz): $F=E-T S=-k T \ln Z \quad, \quad \Delta F \leq 0$
Black body radiation: Stefan's Law for power emitted
$\sigma A T^{4}$
Density of states in three dimensions

$$
D(k)\left(\equiv \frac{\mathrm{d} n}{\mathrm{~d} k}\right)=\frac{V k^{2}}{2 \pi^{2}}
$$

Fermi-Dirac distribution

$$
\frac{1}{\mathrm{e}^{\epsilon / k T}+1}
$$

Bose-Einstein distribution

$$
\frac{1}{\mathrm{e}^{\epsilon / k T}-1}
$$

Fermi momentum for spin-half particles at zero temperature

$$
p_{F}=\hbar\left(6 \pi^{2} n\right)^{1 / 3}
$$

Bose-Einstein condensation Approximate criterion for the critical temperature for Bose-Einstein condensation

$$
\lambda_{T} \approx n^{-1 / 3}
$$

### 6.1 Selected problems

1. The filament of a 100 W lamp has a radius of $12 \mu \mathrm{~m}$ and is 0.3 m long. Estimate its working temperature.
Answer: 3000 K
2.     * Estimate the temperature of a tumbling asteroid at a distance of 350 million km from the Sun. Assume that the surface temperature of the Sun is 6000 K and that the asteroid is a black-body.
Answer: 190 K
3.     * A mole of an ideal diatomic gas is compressed adiabatically by a factor of 2 in volume. How much work is required if it is initially at STP? Answer: 1.81 kJ
4. A mole of ideal gas initially at STP is compressed till the volume has halved but the temperature is unchanged. This is done in two different ways:
a) isothermal compression;
b) isobaric compression followed by isochoric heating.

Draw these two processes on a $P V$ plot, and calculate the work done in each case. Comment on the result in the light of the first law of thermodynamics.
Answer: Work done is (a) 1.57 kJ , (b) 1.13 kJ
5. One 20th of a mole of ideal gas is confined within a volume of 1 litre. A partition is removed to allow it to expand freely into a vacuum, so that the final volume is 2 litres; no heat is exchanged with the surroundings during the process. What are the changes in energy, temperature and entropy of the gas?
Answer: $\Delta S=0.288 \mathrm{~J} \mathrm{~K}^{-1}$
6. Calculate the change in entropy of 0.5 kg of ice at $0^{\circ} \mathrm{C}$ when it is heated to steam at $100^{\circ} \mathrm{C}$. (The latent heats of fusion and evaporation are $3.40 \times 10^{5} \mathrm{Jkg}^{-1}$ and $2.26 \times 10^{6} \mathrm{~J} \mathrm{~kg}^{-1}$ respectively. The specific heat of water is $4200 \mathrm{~J} \mathrm{~kg}^{-1} \mathrm{~K}^{-1}$ ). Answer: $4.30 \times 10^{3} \mathrm{JK}^{-1}$.
7. * Two identical bodies of constant specific heat are at temperatures of $0^{\circ} \mathrm{C}$ and $100^{\circ} \mathrm{C}$. What is the lowest temperature to which both bodies can be brought together in equilibrium by transferring heat from the hotter to the colder body by means of a reversible heat engine? Answer: 319 K
8. * A perfectly reversible heat pump heats a building at $20^{\circ} \mathrm{C}$ by taking heat from the atmosphere at $5^{\circ} \mathrm{C}$. If the heat pump is run by an electric motor whose efficiency is $80 \%$, what is the cost of 1 kWhr of heat supplied? ( 1 kWhr of electricity costs 9 p .) Answer: 0.6p
9. Estimate the steady-state cost per week of running a domestic electric refrigerator. Assume that room temperature is $20^{\circ} \mathrm{C}$ and the internal temperature is $2^{\circ} \mathrm{C}$. The walls are 15 mm thick, having a total surface area of $8 \mathrm{~m}^{2}$ and a thermal conductivity of $2 \times 10^{-2} \mathrm{~W} \mathrm{~m}^{-1} \mathrm{~K}^{-1}$. The cost of electricity is 9 p per kWhr.
Answer: 19p
10. * A two-level system has an energy splitting $\epsilon$ between the upper and lower state. Show that at high temperatures the heat capacity per system varies as $1 / T^{2}$.
11. Estimate the density of neutrinos for which the Fermi temperature would be 3 K , assuming that neutrinos have a rest mass of $10 \mathrm{eV} / c^{2}$ Answer: $1.63 \times 10^{15} \mathrm{~m}^{-3}$ (assumes two spin states)
12. * In the original experiment showing Bose-Einstein condensation of Rubidium 87, 20,000 atoms were trapped and the condensate appeared as the temperature was lowered to 170 nK . Estimate the size of the condensate cloud.
Answer: Of order $20 \mu \mathrm{~m}$ across

## 7 ELECTRICITY AND MAGNETISM

## Fields

The Lorentz force:

$$
\boldsymbol{F}=q(\boldsymbol{E}+\boldsymbol{v} \times \boldsymbol{B})
$$

Maxwell's equations:

$$
\begin{aligned}
\operatorname{div} \boldsymbol{D} & =\rho_{\mathrm{f}}, \\
\operatorname{curl} \boldsymbol{E} & =-\partial \boldsymbol{B} / \partial t
\end{aligned}
$$

$\operatorname{div} B=0 \quad$,
curl $\boldsymbol{H}=j_{\mathrm{f}}+\partial \boldsymbol{D} / \partial t$
where in vacuum

$$
\boldsymbol{D}=\epsilon_{0} \boldsymbol{E} \quad, \quad \boldsymbol{B}=\mu_{0} \boldsymbol{H}
$$

The charge and current are the "free" charge and current.
In linear media, Maxwell's equations hold for smoothly varying "local averaged" fields where the subsidiary $\boldsymbol{D}$ and $\boldsymbol{H}$ fields are now given by

$$
\boldsymbol{D}=\epsilon_{0}(\boldsymbol{E}+\boldsymbol{P})=\epsilon \epsilon_{0} \boldsymbol{E} \quad, \quad \boldsymbol{B}=\mu_{0}(\boldsymbol{H}+\boldsymbol{M})=\mu \mu_{0} \boldsymbol{H}
$$

In addition, for conductors we have Ohm's law:

$$
j=\sigma E
$$

Field energy density:

$$
\frac{E}{V}=\frac{1}{2}(\boldsymbol{D} \cdot \boldsymbol{E}+\boldsymbol{B} \cdot \boldsymbol{H})
$$

Boundary conditions :
$\boldsymbol{B}_{\perp}$ and $\boldsymbol{E}_{\|}$are continuous.
$\boldsymbol{D}_{\perp}$ and $\boldsymbol{H}_{\|}$are continuous in absence of free surface charges and currents. Integral forms of Maxwell's equations:

$$
\begin{array}{cl}
\text { Gauss's theorem } & \int \boldsymbol{E} \cdot \mathrm{d} \boldsymbol{S}=Q / \epsilon_{0}, \\
\text { Faraday's law } & \mathcal{E}=-\frac{\mathrm{d} \Phi}{\mathrm{~d} t} \equiv-\frac{\mathrm{d}}{\mathrm{~d} t}\left[\int \boldsymbol{B} \cdot \mathrm{~d} \boldsymbol{S}\right]
\end{array}
$$

and provided the displacement current vanishes

$$
\text { Ampere's law } \quad \oint_{C} \boldsymbol{H} \cdot \mathrm{~d} \boldsymbol{l}=I_{\mathrm{enc}}
$$

where $I_{\text {enc }}$ is the current through the area enclosed by $C$.

## Circuits

Resistors, capacitors and inductors:

$$
V=I R, \quad V=Q / C, \quad V=L \mathrm{~d} I / \mathrm{d} t
$$

The power dissipated by a resistor is

$$
W=I V=I^{2} R=V^{2} / R
$$

The energy stored in a capacitor or an inductor is

$$
E=C V^{2} / 2 \text { or } E=L I^{2} / 2, \quad \text { respectively. }
$$

AC circuits Here $V=I Z$ where individual impedances are given by

$$
Z_{R}=R \quad, \quad Z_{L}=i \omega L, \quad Z_{C}=1 / i \omega C
$$

Like resistances, impedances in series and in parallel combine as

$$
Z_{\mathrm{ser}}=Z_{1}+Z_{2}, \quad \frac{1}{Z_{\mathrm{par}}}=\frac{1}{Z_{1}}+\frac{1}{Z_{2}}
$$

and the average power dissipated in a load circuit is

$$
W=I_{\mathrm{rms}}^{2}|Z| \cos \theta=I_{\mathrm{rms}} V_{\mathrm{rms}} \cos \theta
$$

where $I_{\mathrm{rms}}=I_{\max } / \sqrt{2}$ etc.. For a resistor, this reduces to the familiar result

$$
W=V_{\mathrm{rms}} I_{\mathrm{rms}}=V_{\mathrm{rms}}^{2} / R=I_{\mathrm{rms}}^{2} R
$$

Resonant frequency of an "LC-circuit":

$$
\omega^{2}=1 / L C
$$

### 7.1 Selected problems

1.     * When a beam of electrons is passed through a region where there are simultaneously present an electric field of $10^{3} \mathrm{~V} \mathrm{~m}^{-1}$ and an orthogonal magnetic field of $10^{-3} \mathrm{~T}$ it is found that the electrons are not deflected. When the electric field is absent the electrons move in a circle of radius 5.7 mm . Calculate the velocity of the electrons and their charge to mass ratio. Answer: $10^{6} \mathrm{~m} \mathrm{~s}^{-1} ; 1.75 \times 10^{11} \mathrm{C} \mathrm{kg}^{-1}$
2. A sulphur sphere of radius 0.1 m and relative permittivity 3.4 is uniformly charged throughout its volume to a charge density of $10^{-5} \mathrm{C} \mathrm{m}^{-3}$. What is the electric field at a point 0.05 m from the centre?
Answer: $5.5 \times 10^{3} \mathrm{~V} \mathrm{~m}^{-1}$
3. Find the smallest radius of curvature that can be used for the corners of a conductor charged to $6 \times 10^{5} \mathrm{~V}$ if breakdown is avoided when the dielectric strength of the air is $3 \times 10^{6} \mathrm{~V} \mathrm{~m}^{-1}$. Answer: 0.2 m
4.     * Assuming that the largest electric field that can be sustained in air is $10^{6} \mathrm{~V} \mathrm{~m}^{-1}$, calculate the maximum electric energy density that can be created in air. Answer: $4.4 \mathrm{~J} \mathrm{~m}^{-3}$
5.     * An infinite metal plate is kept at zero potential. A charge $A$ of 128 coulombs is positioned above the plate, and another charge of 9 coulombs is held half-way between $A$ and the plate. Show that $A$ is in equilibrium.
6. A round, straight copper wire of radius 1 mm carries a steady current of 1 A . Find and sketch the resulting magnetic field both inside and outside the wire.
Answer: $0.2 r$ tesla and $2 \times 10^{-7} / r$ tesla, where $r$ is in metres
7.     * A wire loop of resistance $0.1 \Omega$ has an area of $2.5 \times 10^{-3} \mathrm{~m}^{2}$. It is initially in a uniform magnetic field of 1 T , which is normal to the plane of the loop. How much charge flows round the loop when it is removed to a region of zero magnetic field?
Answer: 25 mC
8.     * A ferromagnetic bar with a length of 0.5 m and a relative permeability of 1000 , is bent into a C shape leaving a 2 mm parallel gap between the bar ends. A 2000-turn coil wound on the bar carries a current of 0.2 A . What is the magnetic field in the gap?
Answer: 0.20 T
9. Define what is meant by the polarisation of an electromagnetic wave. State how two linearly polarised plane waves can be combined to form a circularly polarised wave.
10. A $10 \mu \mathrm{~F}$ capacitor is charged to a voltage $V$ and its terminals are then connected together through a 5 cm length of constantan wire of diameter 0.05 mm . What is the maximum value of $V$ which will not result in fusing the wire?
Assume that the electrical time constant is so short that heat losses may safely be neglected.
Specific heat of constantan $=420 \mathrm{~J} \mathrm{~kg}^{-1} \mathrm{~K}^{-1}$,
Density of constantan $=8880 \mathrm{~kg} \mathrm{~m}^{-3}$,
Melting point of constantan $=1560 \mathrm{~K}$.
Answer: 305 V
11.     * A toroidal coil is made by winding a length of wire uniformly round a non-magnetic insulating thin ring. A $10 \mu \mathrm{~F}$ capacitor charged to 400 V is discharged through the coil. Find the peak value of the magnetic field, given that the volume enclosed by the coil is $100 \mathrm{~cm}^{3}$. Assume that the resistance of the coil is negligible Answer: 0.142 T
12. A long solenoid, wound with 10,000 turns per metre, is bent to form a toroidal coil, enclosing an air volume of $50 \mathrm{~cm}^{3}$. What is its self-inductance? Answer: 6.28 mH
13. A 75 W non-inductive light bulb is designed to run from an ac supply of 120 V rms to 50 Hz . If the only supply available is 240 V rms show that the bulb can be run at the correct power by placing either a resistance $R$, or a capacitor $C$ in series with it. Find the values of $R$ and $C$ and the power drawn from the supply in each case.
Answer: $192 \Omega ; 150 \mathrm{~W} ; 9.6 \mu \mathrm{~F} ; 75 \mathrm{~W}$
14. A circuit has a capacitance $C$, a resistance $R$ and an inductance $L$ in series. The capacitor is initially charged to a voltage $V_{0}$ at time $t=0$ and then a switch is closed to complete the circuit. Given that $R \ll\left|Z_{C}\right|$ and $R \ll\left|Z_{L}\right|$, draw a graph to show how the voltage across the resistor varies with time.

8 WAVES AND OPTICS

## Doppler shift

$$
\begin{aligned}
\text { Nonrelativistic } & \frac{\delta \lambda}{\lambda}=\frac{v}{c}, \\
\text { Relativistic (colinear) } & \frac{\lambda^{\prime}}{\lambda}=\left(\frac{c+v}{c-v}\right)^{1 / 2}
\end{aligned}
$$

Beat frequency:

Phase and group velocity

$$
f_{\text {beat }}=\left|f_{1}-f_{2}\right|
$$

$$
v_{p}=\frac{\omega}{k}, \quad v_{g}=\frac{d \omega}{d k}
$$

Width of a wave packet

$$
\Delta x \Delta k \gtrsim 1
$$

which gives the uncertainty principle $\Delta x \Delta p \gtrsim \hbar$ on substituting $p=k \hbar$. Refractive index

$$
n=\frac{c}{v_{p}}=\frac{\lambda_{0}}{\lambda},
$$

where $\lambda_{0}$ is the wavelength in vacuum.
Snell's law

$$
\frac{\sin \theta_{1}}{\sin \theta_{2}}=\frac{c_{1}}{c_{2}}=\frac{n_{2}}{n_{1}}
$$

The lens formula

$$
\frac{1}{u}+\frac{1}{v}=\frac{1}{f}
$$

Constructive interference In vacuo or a homogeneous medium the condition is

$$
\text { Path difference }=n \lambda
$$

More generally, this result must be modified to take account of the change of wavelength for different refractive indices (see above), and the phase change of $\pi$ which occurs when light is reflected at a boundary with a less dense medium.
Resolving power: diffraction limit

$$
\delta \theta \simeq 1.22 \lambda / d
$$

Here $\delta \theta$ is the angular separation of the peak and first minimum of the diffraction pattern (Rayleigh criterion) so that the full angular width induced in a parallel beam on passing through an aperture is $2 \delta \theta$.

### 8.1 Selected problems

1. An observer standing on a railway platform hears the whistle of an approaching train change its pitch as it passes by from 880 Hz to 720 Hz . What is the speed of the train, given that the speed of sound in air is $330 \mathrm{~m} \mathrm{~s}^{-1}$ ?
Answer: $33 \mathrm{~m} \mathrm{~s}^{-1}$
2.     * The second resonance of an organ pipe open at both ends has the same frequency as the second resonance of a pipe 0.3 m in length open at one end only. What is the length of the open pipe?
Answer: 0.4 m
3. An empty room with sound-reflecting walls has a volume of $50 \mathrm{~m}^{3}$. Estimate the number of (resonant) acoustic modes with frequencies between 440 Hz and 880 Hz . You may assume that the speed of sound in air is $330 \mathrm{~m} \mathrm{~s}^{-1}$.
Answer: ~ 3470
4.     * The phase velocity of surface waves of small wavelength $\lambda$ on a deep liquid is given by

$$
v_{p}^{2}=\frac{2 \pi S}{\lambda \rho}
$$

where $S$ is the surface tension and $\rho$ is the density. Determine the ratio of the group and phase velocities.
Answer: $v_{g} / v_{p}=3 / 2$
5. The group velocity of waves in a medium is found to be twice the phase velocity. Find the relation between the angular frequency $\omega$ and the wave number $k$ in the medium. Answer: $\omega=c k^{2}$
6. * Two guitars have been tuned independently and produce beats when the A-strings are played together.
One guitar is tuned correctly so the A-string oscillates at a frequency of 110 Hz . If the beat frequency is $\frac{3}{4} \mathrm{~Hz}$, what are the possible frequencies of the other A-string vibration?
The frequency of oscillation of a string is proportional to the square root of the string tension. By what fraction must the tension in the out of tune guitar string be changed to bring it in tune?
Answer: 0.014
7. The note from a tuning fork of unknown frequency makes three beats per second with the note from a standard fork of frequency 440 Hz . The beat frequency increases when a small piece of wax is put on a prong of the first (unknown-frequency) fork. What is the frequency of this fork?
Answer: 437 Hz
8. Two laser beams with powers of $100 \mu \mathrm{~W}$ and $1 \mu \mathrm{~W}$ are combined so that they interfere on the surface of a detector. If the frequency of one beam is slightly different from the other, what is the magnitude of the power modulation at the detector?
Answer: 81-121 $\mu \mathrm{W}$
9. With the assistance of a sketch, explain how a mirage is formed.
10. * In normal use, is the image formed by a magnifying glass real or virtual? For an observer with a near point at 25 cm from the eye, what is the greatest magnification that can be obtained with a magnifying glass of focal length 10 cm and where should the object be placed to achieve it?
Answer: 3.5
11. * It is required to broadcast a radio signal at 3000 MHz from the Moon to the whole of the nearside of the Earth. What diameter of antenna should be used? Take the distance of the Moon from the Earth to be $400,000 \mathrm{~km}$.
Answer: 7.6 m
12. At what distance does diffraction limit the reading of a car registration number plate? Answer: Of order 100 m
13. * A diffraction grating has 10,000 lines in 2 cm . What is the angular separation, in second order, of a doublet of wavelengths $6438.0 \AA$ and $6438.1 \AA$, if the grating is used with normal incidence?
Answer: $1.3 \times 10^{-5} \mathrm{rad}$
14. A pair of narrow parallel slits illuminated by monochromatic light of wavelength 500 nm produces interference fringes on a screen. When one of the slits is covered by a thin film of transparent material of refractive index 1.60, the zero order bright fringe moves to the position previously occupied by a bright fringe of the 15th order. What is the thickness of the film? How might the zero order fringe be identified in practice? Answer: $12.5 \mu \mathrm{~m}$
15. * A plane parallel, monochromatic light source of wavelength 650 nm is used to illuminate two parallel slits of equal width, whose centres are separated by 0.15 cm . The fifth order of interference is the first order not observed. How wide are the slits?
Answer: 0.03 cm
16. Unpolarised light of intensity $I_{0}$ is incident on three perfect linear polarisers, placed in series behind each other. The transmission axis, T , of the first is vertical, the second has its transmission axis at an angle $\theta$ to the vertical and the third has its transmission axis at $60^{\circ}$ to the vertical.


The second polariser is initially vertical $(\theta=0)$ and is then rotated. At what values of $\theta$ will the final intensity be zero? What is the general expression for the light intensity as a function of $\theta$ ?

## PHYS30010

## THREE HOURS

A list of constants is enclosed.

## UNIVERSITY OF MANCHESTER

General Physics

## 30th May 2008, 2.00 p.m. -5.00 p.m.

## THREE HOUR CANDIDATES

(Physics, Physics with Astrophysics, Physics with Theoretical Physics, Physics with Technological Physics)
Answer as many questions as you can. Marks will be awarded for your THIRTEEN best answers.

## TWO HOUR CANDIDATES

(Maths/Physics, Physics with Business and Management, Physics with Finance, Physics with Philosophy)
Answer as many questions as you can from questions 1-10 inclusive. Marks will be awarded for your NINE best answers.

Each question is worth 10 marks.

Electronic calculators may be used, provided that they cannot store text.

The numbers are given as a guide to the relative weights of the different parts of each question.

1. What is the resolving power of the human eye at a wavelength of 500 nm . (Assume that the diameter of the pupil is 5 mm .)

What is the diameter of a radio telescope with the same resolution but operating at a frequency of 10 GHz ?
2. An electron with kinetic energy of 5 eV moves in a circular orbit of radius 1 mm in a magnetic field. What is the magnitude and direction of the field?
3. A frictionless chain hangs on two inclined planes, as indicated in the drawing. Prove that the chain is in equilibrium, i.e. will not slip either to the left or to the right.

4. Compute the wavelength of a photon emitted when an electron transits from an energy level with $n=167$ to that with $n=166$ of the Hydrogen atom.
5. Write down an expression for the pressure in an isothermal atmosphere at temperature T as a function of the height Z , the surface pressure $\mathrm{p}_{o}$ and the average molecular weight m of the atmospheric constituants.

Calculate the scale height (at which the pressure has fallen to $1 / \mathrm{e}$ of its surface value) of the Earth's atmosphere, given that $m=29$ a.m.u.
6. A beam of 0.27 eV neutrons is directed onto the surface of a crystal with interatomic spacing $2 \times 10^{-10} \mathrm{~m}$. What is the angle between the surface and the incident beam at which strong diffraction will be observed?
7. A parallel beam of light of diameter 1 mm enters a concave lens of focal length 100 mm . The beam expands and is at a diameter of 5 mm where it enters a convex lens of focal length 100 mm . The beam is then brought to a focus.

Sketch a typical ray path through the lenses. What is the distance between the two lenses? What is the distance from the convex lens to the focus of the beam?
8. Write down an expresion for the entropy $S$ of a system in terms of the number of quantum microstates $\Omega$.

A system is initially in a state 1 and changes to a state 2 such that the number of quantum states available to the system is increased by a factor $\mathrm{e}^{10^{23}}$. Calculate the increase in entropy of the system.
9. In 1987, a supernova exploded at a distance of $50 \mathrm{kpc}\left(1.54510^{21} \mathrm{~m}\right)$. Twelve neutrinos were detected on Earth following this eruption. Assume they arrived 2 hours after the explosion was seen. Calculate (i) $v / c$ of the neutrinos, and (ii) the travel time of the neutrinos in their own frame of rest: give your answer in years.

Hint: express $v / c$ in the form $1-\delta$.
10. For a particle of rest mass $m$, derive a formula for the total energy that would be required in order for it to resolve a structure of size $\lambda$. Estimate this for the case of resolving the internal structure of a nucleus using (i) an electron and (ii) a proton.
11. By treating them both as black bodies estimate the relative amounts of energy radiated by the Sun and the Earth.

At what wavelength should a distant observer choose to make measurements in order to maximise the visibility of the Earth against the Sun?
12. The wavefunctions of two states of an atom are given by

$$
\psi_{1}(r, \theta, \phi)=R(r) \cos \theta \cos \phi, \quad \psi_{2}(r, \theta, \phi)=R(r) \cos \theta \sin \phi .
$$

Show that these are not eigenfunctions of the angular momentum operator $\hat{L}_{z}=-i \hbar \frac{\partial}{\partial \phi}$. By considering linear combinations of these two states construct two states with definite angular momentum in the $z$-direction.
13. Show that if two springs with spring constants $k_{1}$ and $k_{2}$ are connected in series, the net spring constant of the combination is given by

$$
\frac{1}{k}=\frac{1}{k_{1}}+\frac{1}{k_{2}}
$$

14. A reversible heat pump heats a building to $25^{\circ} \mathrm{C}$ using heat from a river at $5^{\circ} \mathrm{C}$. Derive the efficiency (performance coefficient) for such a pump and calculate the number of Joules of heat produced in the building by the expenditure of 1 J in the pump.
15. Sound waves in a solid obey the dispersion relation:

$$
\omega=a \sin \left(\frac{k}{k_{0}}\right)
$$

where $\omega$ is the angular frequency, $k$ the wavenumber and $a$ and $k_{o}$ are constants. Show that

$$
k^{2} V_{p}^{2}+k_{o}^{2} V_{g}^{2}=a^{2}
$$

where $V_{p}$ and $V_{g}$ are the phase and group velocity of the waves.

END OF EXAMINATION PAPER

PHYSICAL CONSTANTS AND CONVERSION FACTORS

| SYMBOL | DESCRIPTION | NUMERICAL VALUE |
| :---: | :---: | :---: |
| c | Velocity of light in vacuum | $299792458 \mathrm{~m} \mathrm{~s}^{-1}$, exactly |
| $\mu_{0}$ | Permeability of vacuum | $4 \pi \times 10^{-7} \mathrm{~N} \mathrm{~A}^{-2}$, exactly |
| $\epsilon_{0}$ | Permittivity of vacuum where $c=1 / \sqrt{\epsilon_{0} \mu_{0}}$ | $8.854 \times 10^{-12} \mathrm{C}^{2} \mathrm{~N}^{-1} \mathrm{~m}^{-2}$ |
| $h$ | Planck constant | $6.626069 \times 10^{-34} \mathrm{~J} \mathrm{~s}$ |
| $\hbar$ | $h / 2 \pi$ | $1.054572 \times 10^{-34} \mathrm{~J} \mathrm{~s}$ |
| G | Gravitational constant | $6.674 \times 10^{-11} \mathrm{~m}^{3} \mathrm{~kg}^{-1} \mathrm{~s}^{-2}$ |
| $e$ | Elementary charge | $1.6021765 \times 10^{-19} \mathrm{C}$ |
| eV | Electronvolt | $1.6021765 \times 10^{-19} \mathrm{~J}$ |
| $\alpha$ | Fine-structure constant, $e^{2} /\left(4 \pi \epsilon_{0} \hbar c\right)$ | 1/137.036 |
| $m_{e}$ | Electron mass | $9.109382 \times 10^{-31} \mathrm{~kg}$ |
| $m_{e} c^{2}$ | Electron rest-mass energy | 0.510999 MeV |
| $\mu_{B}$ | Bohr magneton, eћ/ ( $\left.2 m_{e}\right)$ | $9.27401 \times 10^{-24} \mathrm{JT}^{-1}$ |
| $R_{\infty}$ | Rydberg energy, $\alpha^{2} m_{e} c^{2} / 2$ | 13.60569 eV |
| $a_{0}$ | Bohr radius, (1/ $\alpha) \hbar /\left(m_{e} c\right)$ | $0.5292177 \times 10^{-10} \mathrm{~m}$ |
| Å | Angstrom | $10^{-10} \mathrm{~m}$, exactly |
| $m_{p}$ | Proton mass | $1.672622 \times 10^{-27} \mathrm{~kg}$ |
| $m_{p} c^{2}$ | Proton rest-mass energy | 938.27203 MeV |
| $m_{n} c^{2}$ | Neutron rest-mass energy | 939.56536 MeV |
| $\mu_{N}$ | Nuclear magneton, eћ/ $\left.2 m_{p}\right)$ | $5.05078 \times 10^{-27} \mathrm{~J} \mathrm{~T}^{-1}$ |
| fm | Femtometre or fermi | $10^{-15} \mathrm{~m}$, exactly |
| b | Barn | $10^{-28} \mathrm{~m}^{2}$, exactly |
| $u$ | Atomic mass unit, $m\left({ }^{12} \mathrm{C}\right.$ atom)/12 | $1.660539 \times 10^{-27} \mathrm{~kg}$ |
| $N_{A}$ | Avogadro constant, atoms in gram mol | $6.022142 \times 10^{23} \mathrm{~mol}^{-1}$ |
| $T_{t}$ | Triple-point temperature | 273.16 K |
| $k$ | Boltzmann constant | $1.38065 \times 10^{-23} \mathrm{~J} \mathrm{~K}^{-1}$ |
| $R$ | Molar gas constant, $N_{A} k$ | $8.3145 \mathrm{~J} \mathrm{~mol}^{-1} \mathrm{~K}^{-1}$ |
| $\sigma$ | Stefan-Boltzmann constant, $\left(\pi^{2} / 60\right) k^{4} /\left(\hbar^{3} c^{2}\right)$ | $5.6704 \times 10^{-8} \mathrm{~W} \mathrm{~m}^{-2} \mathrm{~K}^{-4}$ |
| $M_{E}$ | Mass of the Earth | $5.9742 \times 10^{24} \mathrm{~kg}$ |
| $R_{E}$ | Mean radius of the Earth | $6.3675 \times 10^{6} \mathrm{~m}$ |
| $g$ | Standard acceleration of gravity | $9.80665 \mathrm{~m} \mathrm{~s}^{-1}$, exactly |
| atm | Standard atmosphere | 101325 Pa , exactly |
| $M_{\odot}$ | Solar mass | $1.988435 \times 10^{30} \mathrm{~kg}$ |
| $R$. | Solar radius | $6.955 \times 10^{8} \mathrm{~m}$ |
| $L_{\odot}$ | Solar luminosity | $3.846 \times 10^{26} \mathrm{~W}$ |
| $T_{\odot}$ | Solar effective temperature | 5780 K |
| AU pc | Astronomical unit, mean Earth-Sun distance | $1.495978 \times 10^{11} \mathrm{~m}$ |
|  | Parsec | $3.085678 \times 10^{16} \mathrm{~m}$ |
|  | Year | $3.1536 \times 10^{7}$ s |

## Exponentials and Logarithms

$$
\begin{aligned}
& \mathrm{e}^{\ln x}=x, \quad \ln (\mathrm{e})=1, \quad \ln \left(\mathrm{e}^{a}\right)=a, \quad \ln \left(x^{a}\right)=a \ln x, \quad \log _{a} x=\ln x / \ln a \\
& \mathrm{e}^{a} \mathrm{e}^{b}=\mathrm{e}^{a+b}, \quad \ln (x y)=\ln x+\ln y, \quad \ln (x / y)=\ln x-\ln y .
\end{aligned}
$$

## Complex Numbers

The imaginary unit $i$ is defined by $i^{2}=-1$; the symbol " $j$ " is sometimes used as well. A complex number can be written in terms of real and imaginary part, or in terms of the modulus $r$ and angle $\theta$ (de Moivre's theorem),

$$
r \equiv z=x+i y=r \mathrm{e}^{i \theta}=r[\cos \theta+i \sin \theta], \quad \theta=\arctan (y / x) .
$$

Under complex conjugation we take $i \rightarrow-i$,

$$
z^{*}=x-i y=r \mathrm{e}^{-i \theta}=r[\cos \theta-i \sin \theta] .
$$

The modulus of $z$ is the real number

$$
|z|=\sqrt{z^{*} z}=\sqrt{x^{2}+y^{2}} .
$$

We can define the sine and cosine in terms of complex numbers by

$$
\cos \theta=\frac{1}{2}\left[\mathrm{e}^{i \theta}+\mathrm{e}^{-i \theta}\right], \quad \sin \theta=\frac{1}{2 i}\left[\mathrm{e}^{i \theta}-\mathrm{e}^{-i \theta}\right] .
$$

## Trigonometry

$\sin ^{2} x+\cos ^{2} x=1$,
$\sin (x \pm y)=\sin x \cos y \pm \cos x \sin y$,
$\cos (x \pm y)=\cos x \cos y \mp \sin x \sin y$,
This allows us to derive the following equations
$\sin x \cos y=\frac{1}{2}(\sin (x+y)+\sin (x-y))$,
$\cos x \cos y=\frac{1}{2}(\cos (x+y)+\cos (x-y))$,
$\sin x \sin y=\frac{1}{2}(-\cos (x+y)+\cos (x-y))$.
$\sin x \pm \sin y=2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y)$,
$\cos x+\cos y=2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)$,
$\cos x-\cos y=-2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y)$.
$\sin 2 x=2 \sin x \cos x$,
$\cos 2 x=\cos ^{2} x-\sin ^{2} x=2 \cos ^{2} x-1=1-2 \sin ^{2} x$.

## Hyperbolic functions

$\cosh x=\frac{1}{2}\left[\mathrm{e}^{x}+\mathrm{e}^{-x}\right]=\cos (i x), \quad \sinh x=\frac{1}{2}\left[\mathrm{e}^{x}-\mathrm{e}^{-x}\right]=-i \sin (i x)$.

## Power Series

Geometric Series

$$
\begin{aligned}
& 1+x+x^{2}+\ldots+x^{n}=\frac{1-x^{n+1}}{1-x} \\
& \sum_{j=1}^{\infty} x^{n}=\frac{1}{1-x}, \quad|x|<1
\end{aligned}
$$

Binomial expansion

$$
(1+x)^{n}=1+n x+\frac{n(n-1)}{2!} x^{2}+\frac{n(n-1)(n-2)}{3!} x^{3}+\ldots
$$

$$
\binom{n}{m}=\frac{n!}{m!(n-m)!}
$$

Here $n$ is not necessarily an integer. Taylor series of $f(x)$ about $x=x_{0}$ :

$$
f(x)=f\left(x_{0}\right)+\left.\left(x-x_{0}\right) \frac{\mathrm{d} f}{\mathrm{~d} x}\right|_{x=x_{0}}+\left.\frac{1}{2}\left(x-x_{0}\right)^{2} \frac{\mathrm{~d}^{2} f}{\mathrm{~d} x^{2}}\right|_{x=x_{0}}+\ldots
$$

Some Standard MacLaurin series (Taylor series, but $x_{0}=0$ )

$$
\begin{aligned}
& \mathrm{e}^{x}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\ldots \quad x \in \mathbb{R}, \\
& \cos x=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}+\ldots \quad x \in \mathbb{R}, \\
& \sin x=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}+\ldots \quad x \in \mathbb{R}, \\
& \ln (1 \pm x)= \pm x-\frac{x^{2}}{2} \pm \frac{x^{3}}{3} \ldots \quad|x|<1, \\
& (1 \pm x)^{-1}=1 \pm x+x^{2} \pm x^{3} \ldots \quad|x|<1 .
\end{aligned}
$$

L'Hôpital's rule: if $f\left(x_{0}\right)=g\left(x_{0}\right)=0$,
$\lim _{x \rightarrow x_{0}} \frac{f(x)}{g(x)}=\lim _{x \rightarrow x_{0}} \frac{\mathrm{~d} f / \mathrm{d} x}{\mathrm{~d} g / \mathrm{d} x}$

## Differentiation and integration

Standard derivatives ( $a$ is a constant)

| $f(x)$ | $x^{n}$ | $\mathrm{e}^{a x}$ | $\ln x$ |
| :--- | :--- | :--- | :--- |
| $\mathrm{~d} f / \mathrm{d} x$ | $n x^{n-1}$ | $a \mathrm{e}^{a x}$ | $1 / x$ |
|  |  |  |  |
| $f(x)$ | $\sin x$ | $\cos x$ | $\tan x$ |
| $\mathrm{~d} f / \mathrm{d} x$ | $\cos x$ | $-\sin x$ | $\sec ^{2} x$ |
| $f(x)$ | $\sinh x$ | $\cosh x$ | $\tanh x$ |
| $\mathrm{~d} f / \mathrm{d} x$ | $\cosh x$ | $\sinh x$ | $\operatorname{sech}^{2} x$ |
|  |  |  |  |
| $f(x)$ | $\sin ^{-1}(x / a)$ | $\cos ^{-1}(x / a)$ | $\tan ^{-1}(x / a)$ |
| $\mathrm{d} f / \mathrm{d} x$ | $1 / \sqrt{a^{2}-x^{2}}$ | $-1 / \sqrt{a^{2}-x^{2}}$ | $a /\left(a^{2}+x^{2}\right)$ |
| $f(x)$ | $\sinh ^{-1}(x / a)$ | $\cosh ^{-1}(x / a)$ | $\tanh ^{-1}(x / a)$ |
| $\mathrm{d} f / \mathrm{d} x$ | $1 / \sqrt{a^{2}+x^{2}}$ | $1 / \sqrt{x^{2}-a^{2}}$ | $a /\left(a^{2}-x^{2}\right)$ |

All of these can be used to calculate integrals. The only one that has a more generic expression is

$$
\int \frac{1}{x} \mathrm{~d} x=\ln |x|+c .
$$

Note the appearance of the constant of integration.

## Product rule and integration by parts

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x}[f(x) g(x)]=\frac{\mathrm{d} f(x)}{\mathrm{d} x} g(x)+f(x) \frac{\mathrm{d} g(x)}{\mathrm{d} x} \\
& \int f(x) \frac{\mathrm{d} g(x)}{\mathrm{d} x} \mathrm{~d} x=f(x)\left(g(x)-\int \frac{\mathrm{d} f(x)}{\mathrm{d} x} g(x) \mathrm{d} x\right.
\end{aligned}
$$

Chain rule and integration by substitution

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x} f(g(x))=\frac{\mathrm{d} f(g)}{\mathrm{d} g} \frac{\mathrm{~d} g(x)}{\mathrm{d} x} \\
& \int F(g(x)) \frac{\mathrm{d} g(x)}{\mathrm{d} x} \mathrm{~d} x=\int F(g) \mathrm{d} g
\end{aligned}
$$

## Partial Derivatives

Given a function of more than one variable, e.g., $f=f(x, y)$, we have

$$
\mathrm{d} f=\left(\frac{\partial f}{\partial x}\right)_{y} \mathrm{~d} x+\left(\frac{\partial f}{\partial y}\right)_{x} \mathrm{~d} y
$$

If $x=x(r, \phi)$ and $y=y(r, \phi)$ we get the chain rule

$$
\left(\frac{\partial f}{\partial r}\right)_{\phi}=\left(\frac{\partial f}{\partial x}\right)_{y}\left(\frac{\partial x}{\partial r}\right)_{\phi}+\left(\frac{\partial f}{\partial y}\right)_{x}\left(\frac{\partial y}{\partial r}\right)_{\phi}
$$

## Differential equations

The following 3 ODEs can be solved by substituting an exponential solution,

$$
\begin{array}{rl}
\dot{y}=\lambda y & y=A \mathrm{e}^{\lambda t} \\
\ddot{y}=-\omega^{2} y & y=A \cos (\omega t+\phi) \\
\ddot{y}+\gamma \dot{y}+\omega_{0}^{2} y=0 & y=A \mathrm{e}^{-\gamma t / 2} \cos (\omega t+\phi)
\end{array}
$$

In the last equation $\omega=\sqrt{\omega_{0}^{2}-\gamma^{2} / 4}$, and we assume $2 \omega_{0}>|\gamma|$ (underdamped).
Other useful techniques: Separation of variables, and the use of the "integrating factor".

## Vectors

A vector $\boldsymbol{a}$ is defined by its components as

$$
\boldsymbol{a}=\left(a_{x}, a_{y}, a_{z}\right)=a_{x} \hat{\boldsymbol{i}}+a_{y} \hat{\boldsymbol{j}}+a_{z} \hat{\boldsymbol{k}} .
$$

The modulus (length) of $\boldsymbol{a}$ is

$$
|\boldsymbol{a}|(\equiv a)=\sqrt{a_{x}^{2}+a_{y}^{2}+a_{z}^{2}}=\sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} .
$$

Scalar product

$$
\boldsymbol{a} \cdot \boldsymbol{b}=a_{x} b_{x}+a_{y} b_{y}+a_{z} b_{z}=|\boldsymbol{a}||\boldsymbol{b}| \cos \theta
$$

Vector product

$$
\boldsymbol{a} \times \boldsymbol{b}=\left(a_{y} b_{z}-a_{z} b_{y}, a_{z} b_{x}-a_{x} b_{z}, a_{x} b_{y}-a_{y} b_{x}\right)=\left|\begin{array}{ccc}
\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\
a_{x} & a_{y} & a_{z} \\
b_{x} & b_{y} & b_{z}
\end{array}\right|=a b \sin \theta \hat{\boldsymbol{n}} .
$$

The vectors $\boldsymbol{a}, \boldsymbol{b}$ and the unit vector $\hat{\boldsymbol{n}}$ form a right-handed set.

## Scalar Triple Product

$$
\boldsymbol{a} \cdot(\boldsymbol{b} \times \boldsymbol{c})=\boldsymbol{b} \cdot(\boldsymbol{c} \times \boldsymbol{a})=\boldsymbol{c} \cdot(\boldsymbol{a} \times \boldsymbol{b})=\left|\begin{array}{lll}
a_{x} & a_{y} & a_{z} \\
b_{x} & b_{y} & b_{z} \\
c_{x} & c_{y} & c_{z}
\end{array}\right| .
$$

## Vector Triple Product

$\boldsymbol{a} \times(\boldsymbol{b} \times \boldsymbol{c})=\boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c})-\boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$.

## Coordinate systems

Vector coordinates: The position vector is usually called $\boldsymbol{r}$,

$$
\boldsymbol{r} \equiv x \hat{\boldsymbol{i}}+y \hat{\boldsymbol{j}}+z \hat{\boldsymbol{k}}
$$

The radial distance from the origin is then

$$
r=\sqrt{x^{2}+y^{2}+z^{2}}
$$

The direction of $\boldsymbol{r}$ is given by the unit vector $\hat{\boldsymbol{r}}=\boldsymbol{r} / r$.

Plane polar coordinates (in 2D) $(r, \phi)$ or $(r, \theta)$

$$
\begin{aligned}
& r=\sqrt{x^{2}+y^{2}}, \quad \phi=\arctan (y / x) ; \\
& x=r \cos \phi, \quad y=r \sin \phi \\
& \hat{\boldsymbol{r}}=\boldsymbol{r} / r, \quad \hat{\boldsymbol{\phi}}=(-\sin \phi, \cos \phi) \\
& \operatorname{grad}=\hat{\boldsymbol{r}} \partial_{r}+\hat{\boldsymbol{\phi}} \partial_{\phi}, \\
& \mathrm{d} A=r \mathrm{~d} r \mathrm{~d} \phi \quad \text { (Area element) }
\end{aligned}
$$

## Cylindrical polar coordinates $(\rho, \phi, z)$

For $\rho$ and $\phi$ read the 2D plane polar coordinates above.

$$
\begin{aligned}
& \hat{\boldsymbol{\rho}}=(\cos \phi, \sin \phi, 0), \quad \hat{\boldsymbol{\phi}}=(-\sin \phi, \cos \phi, 0) ; \\
& \operatorname{grad}=\hat{\boldsymbol{\rho}} \partial_{\rho}+\hat{\boldsymbol{\phi}} \partial_{\phi}+\hat{\boldsymbol{k}} \partial_{z}, \\
& \mathrm{~d} V=\rho \mathrm{d} \rho \mathrm{~d} \phi \mathrm{~d} z \quad \text { (Volume element })
\end{aligned}
$$

Note: One often meets $r$ instead of $\rho$. Avoid this to reduce confusion.

## Spherical coordinates $(r, \theta, \phi)$

$$
\begin{aligned}
& r=\sqrt{x^{2}+y^{2}+z^{2}}, \quad \phi=\arctan (y / x), \quad \theta=\arctan \left(\sqrt{x^{2}+y^{2}} / z\right) ; \\
& x=r \cos \phi \sin \theta, \quad y=r \sin \phi \sin \theta, \quad z=r \cos \theta ; \\
& \hat{\boldsymbol{r}}=\boldsymbol{r} / r, \quad \hat{\boldsymbol{\phi}}=(-\sin \phi, \cos \phi, 0), \quad \hat{\boldsymbol{\theta}}=(\cos \phi \cos \theta, \sin \phi \cos \theta,-\sin \theta) ; \\
& \operatorname{grad}=\hat{\boldsymbol{r}} \partial_{r}+\hat{\boldsymbol{\phi}} \partial_{\phi}+\hat{\boldsymbol{\theta}} \partial_{\theta}, \\
& \mathrm{d} V=r^{2} \sin \theta \mathrm{~d} r \mathrm{~d} \phi \mathrm{~d} \theta=-r^{2} \mathrm{~d} r \mathrm{~d} \phi \mathrm{~d} \cos \theta \quad \text { (Volume element) }
\end{aligned}
$$

## Vector Calculus

$$
\begin{aligned}
& \operatorname{grad} f=\boldsymbol{\nabla} f=\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
& \begin{aligned}
\operatorname{div} \boldsymbol{\psi}=\boldsymbol{\nabla} \cdot \boldsymbol{\psi}=\frac{\partial \psi_{x}}{\partial x}+\frac{\partial \psi_{y}}{\partial y}+\frac{\partial \psi_{z}}{\partial z} \\
\operatorname{curl} \boldsymbol{\psi}=\boldsymbol{\nabla} \times \boldsymbol{\psi}=\left(\frac{\partial \psi_{z}}{\partial x}-\frac{\partial \psi_{x}}{\partial z}\right) \hat{\boldsymbol{i}}+\left(\frac{\partial \psi_{x}}{\partial y}-\frac{\partial \psi_{y}}{\partial x}\right) \hat{\boldsymbol{j}}+\left(\frac{\partial \psi_{y}}{\partial z}-\frac{\partial \psi_{z}}{\partial y}\right) \hat{\boldsymbol{k}}, \\
\nabla^{2} f=\frac{\partial^{2} f}{\partial x^{2}}+\frac{\partial^{2} f}{\partial y^{2}}+\frac{\partial^{2} f}{\partial z^{2}} \\
\quad=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \frac{\partial}{\partial r} f\right)+\frac{1}{r^{2} \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta} f\right)+\frac{1}{r^{2} \sin ^{2} \theta} \frac{\partial^{2}}{\partial \phi^{2}} f, \\
\operatorname{div} \operatorname{grad} f=\nabla^{2} f, \\
\operatorname{curl} \operatorname{grad} f=0, \\
\operatorname{div} \operatorname{curl} \boldsymbol{\psi}=0, \\
\operatorname{curl} \operatorname{curl} \boldsymbol{\psi}=\operatorname{grad}(\operatorname{div} \boldsymbol{\psi})-\nabla^{2} \boldsymbol{\psi} .
\end{aligned}
\end{aligned}
$$

## Stokes theorem:

$\int_{S}(\boldsymbol{\operatorname { c u r l }} \boldsymbol{F}) \cdot \mathrm{d} \boldsymbol{S}=\oint_{\delta S} \boldsymbol{F} \cdot \mathrm{~d} \boldsymbol{r}$,
( $\delta S$ is the curve bounding the surface $S$.)
Divergence theorem

$$
\int_{V} \operatorname{div} \boldsymbol{F} \mathrm{~d} V=\int_{S} \boldsymbol{F} \cdot \mathrm{~d} \boldsymbol{S}
$$

( $S$ is the surface of the volume $V$ )

